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考慮極值與 VaR 限制之最適資產配置

Optimal Asset Allocation with Extreme Returns and a VaR Constraint

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摘要

許多實證研究顯示資產報酬分配呈左偏和肥尾。本文探討當資產報酬分配呈左偏和肥尾時，對風險管理者資產配置之影響。Basak 與 Shapiro (2001) 是首位將風險限制式 (VaR) 納入效用函數內，再極大化投資人之效用函數而求出最適資產配置。本文依據他們的方法，採用 Gram-Charlier expansion 描述資產報酬左偏和肥尾之特性，探討當資產報酬分配在非常態分配下，其資產配置的變化。對風險管理者而言，最重要的工作就是準確預測損失與發生損失的機率。瞭解資產報酬的型態將有助於準確的預測損失，我們無法降低損失，但可以降低發生損失的機率，本文建議可以降低 α 值 (期末財富損失大於 VaR 之機率) 來達成，而降低 α 值會使期末財富在好的狀態與壞的狀態的財富稍減。

【關鍵字】極值、左偏、肥尾

Abstract

This study investigates how deviations from normality affect asset choices made by risk managers. This study applies the Gram-Charlier expansion for negatively skewed and excess kurtosis. Following Basak and Shapiro (2001), this study examines how negatively skewed and excess kurtosis affects asset allocations when investors manage market-risk exposure using Value-at-Risk-based risk management (VaR-RM). It is important for risk managers to precisely forecast the loss. The analytical results imply that the impact of leptokurtic asset returns is based on the shape of asset returns, and a correct measurement of leptokurtic asset returns is helpful to risk managers seeking to precisely forecast the loss. A risk manager cannot reduce the loss in bad states, but can reduce the value of α, the probability that a loss exceeds VaR, and the agent will suffer from reduced terminal wealth in both the good and bad states.

【Keywords】 Gram-Charlier expansion, value-at-risk-based risk management, leptokurtic asset returns

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A. Introduction

A vast amount of empirical evidence demonstrates that asset returns generally exhibit *leptokurtic behaviors*, i.e., asset returns are negatively skewed and have fat tails (Praetz, 1972; de Vries, 1994; Campbell, Lo, & Mackinlay, 1997). Fama (1965) was the first to demonstrate that stock returns have fat tails. Many reasons account for the leptokurtic behavior of return distributions, for example, volatility changes over time, or extreme events occur. These extreme returns generate high moments differ from the high moments one would obtain with a normal distribution.

One may ask what the impact is of ignoring leptokurtic behavior in asset return models building various financial issues. Tokat, Rachev, and Schwartz (2003) used Stable non-Gaussian distributions to describe the fat tails of asset returns and demonstrated that Stable scenario modeling leads to equity asset allocations up to 26% less than those of acquired by normal scenario modeling. Hales (1997) indicated that an options pricing model with capturing fat tails in underlying asset distributions reduces pricing biases relative to the Gaussian Black-Scholes model for valuing foreign currency options. Bidarkota and McCulloch (2003) showed that accounting for fat tails in the dividends data produces an additional 13% of equilibrium equity returns in a standard consumption-based asset-pricing model.

Fat tails are not important when one is interested only in expected returns. However, when one is interested in measures that describe uncertainty or risk associated with a given asset allocation strategy, the precise shape of distribution tails matters a great deal (Jorion, 1996). In financial markets, small but non-zero probability of extreme events, regardless of how unlikely, can have a significant impact. Stulz (2003) suggested that the failure of Long-Term Capital Management (LTCM) in 1998 had an dramatic affect on risk management beliefs. Two famous financial economists, Robert Merton and Myron Scholes, were LTCM partners. Notably, LTCM was very successful in its first two years (1995, 1996). However, it experienced two consecutive months of huge losses in 1998. The fund losses in August and September of 1998 had a probability of zero, yet they occurred. The fund exceeded the one-week VaR for five consecutive weeks. With independent weekly returns, such an event has probability of 0.0000003 ($= 0.05^5$) of occurring. Stulz proposed that four reasons account for downward bias of VaR estimates. One reason is that extreme returns, the tails of the density function, have higher probabilities than those of the normal distribution. It is likely that when one faces the mere possibility of extreme returns, he will not allocate his wealth in the same manner as one who only cares about the first and second moment.

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Various alternative distributions have been developed to describe these fat tails. For example, Extreme Value Theory (EVT) addresses tail behavior of the asset returns (Embrechts, Kluppelberg, & Mikosch, 1997). This theory investigates the distribution of maximum (minimum) in large samples, thereby determining a distribution's tail shapes. However, information about the central characteristics of the distribution is lost. Stable Paretian distribution is an alternative model that captures fat tails and the whole return distributions (Samorodnitsky & Taqqu, 1994; Rachev & Mittnik, 2000; Tokat et al., 2003). The shortcomings of a Stable Paretian distribution are that densities and distribution functions are not known in a closed form for most Stable distributions and Stable distributions are generally specified by their characteristic functions. Engle's (1982) autoregressive conditional heteroskedasti (ARCH) model and its various generalizations have indicated that the conditional normal or Student's t ARCH process is insufficiently fat-tailed to account for excess kurtosis in data (Nelson, 1991).

An accurate model that describes asset returns is required to measure and manage risk. One can rely on a polynomial expansion of a normal density function, such as the Gram-Charlier expansion. Jarrow and Rudd (1982) first applied the Gram-Charlier expansion to finance. This expansion addresses skewness and kurtosis. The Gram-Charlier expansion accommodates a virtually unlimited class of possible shapes and generates an approximate density function for a standardized random variable. This density function has been applied in Gallant, Hansen and Tauchen (1990), Gallant and Nychka (1987), Gallant and Tauchen (1989), Longstaff (1995), Corrado and Su (1996, 1997), and Jondeau and Rockinger (2001).

Value-at-Risk has become a standard used by financial institutions to measure and manage risk. Value-at-Risk describes the loss that can occur over a given period, at a given confidence level, due to market risk exposure (Jorion, 1996). The VaR at 5 percent is the loss that has a 5 percent probability of being exceeded. The VaR can be biased when one assumes an incorrect distribution for asset returns. Suppose that returns are assumed to be normally distributed, one can have an incorrect VaR as returns in reality follow a leptokurtic distribution. Basak and Shapiro (2001) analyzed optimal portfolio and wealth/consumption policies of utility maximizing investors who must also manage market-risk exposure using VaR. They applied the VaR constraint directly to the utility maximization problem, thereby facilitating investigation of the impact of VaR risk management (VaR-RM) on endogenously determined economic quantities.

This study is interested in the effect of extreme returns or fat tails on the solution of the asset allocation problem encountered by risk managers. Therefore, this study introduces

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Gram-Charlier expansion that allows for negatively skewed and excess kurtosis. This study extends the analytical results obtained by Basak and Shapiro (2001) and investigates how high moments affect asset allocations when investors manage market-risk exposure using VaR-RM. The first two moments correspond to mean and variance. A negative third moment demonstrates that there are more extreme negative realizations than positive realizations. The fourth moment measures the relative concentration of values in the distribution center as compared with those at the tails.

The principal findings are as follows

(1) When risky asset return has negative skewness and excess kurtosis, and the VaR manager allocates their portfolio under the assumption of normal asset return, then the agent will experience an unexpected and great loss when the end-of-horizon state is a bad state.

(2) The agent invests reduced funds in the stock market during intermediate states and increased amounts in the stock market during bad states when confronted with negative skewness and excess kurtosis.

(3) A risk manager cannot decrease losses in bad states, but can decrease the value of α, the probability that a loss exceeds VaR, and the agent will suffer from reduced terminal wealth in both the good and bad states.

It is important for risk managers to precisely forecast the loss. The analytical results imply that the impact of leptokurtic asset returns is based on the shape of asset returns, and a correct measurement of leptokurtic asset returns is helpful to risk managers seeking to precisely forecast the loss.

This work is similar to that of Lucas and Klaassen (1998) and Tokat et al. (2003). Lucas and Klaassen considered a simple one-period asset allocation problem and investigated the effect of extreme returns or fat tails on optimal asset allocations using a shortfall constraint. They utilized the Student-t distribution to characterize the fat tails of asset returns and maximize expected return on the portfolio subject to a VaR constraint. They showed that the VaR of a given portfolio may be underestimated when tail behavior of asset returns is not captured accurately. Tokat et al. (2003) investigated the effects of fat-tailed returns on asset allocation decisions when an investor uses VaR and Conditional VaR as risk measures. The solution methodology comprises a multistage stochastic asset allocation problem with decision rules. The uncertainty is modeled utilizing economic scenarios with Gaussian and stable non-Gaussian distributions. Unlike Lucas and Klaassen (1998) and Tokat et al. (2003), this study focuses on solving the optimal investment problem

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for a VaR risk manager, and examines the effects of extreme returns or fat tails on optimal asset allocations. This study does not analyze expected utility maximization and risk management as mutually exclusive paradigms, but rather directly embeds risk management objectives into a framework that maximizes utility.

This paper is organized as follows. In Section (B), the general economic framework is presented and the approach for modeling returns with a non-normal distribution is described. Section (C) analyzes the asset allocation problem under VaR when asset returns are non-normal. Section (D) presents the numerical illustrations via simplified examples. Section (E) presents the conclusion.

B. Economic Setting

This section formulates an economy. Financial markets where trading takes place continuously between time zero and time T are considered. Consider a frictionless securities market through which two well-established securities are traded. Security B is a local riskless asset and stock index S is a risky asset. Section (A) shows the general assumptions of asset return in most academic financial literature. These assumptions are the same as those described by Basak and Shapiro (2001). Section (B) presents the development a stock index model that allows for negatively skewed and fat-tailed. This model is employed in determining how negatively skewed and excess kurtosis affects asset allocations.

(A) Asset returns with normal distribution

Suppose the short-term interest rate r is constant over time interval [0, T], and the value of a riskless bond denoted by B is assumed to be continuously compounded at rate r; that is,

$$dB(t) = B(t)rdt, \tag{1}$$

or $B(t) = e^{rt}$, with $B(0) = 1$.

The stochastic process for stock index price is

$$dS(t) = S(t)\mu dt + S(t)\sigma d\omega(t) \tag{2}$$

where μ is the drift rate, σ is the stock index volatility and $\omega(t)$ is a standard Brownian motion.

The fraction of wealth invested in stock index at time t is denoted by $\theta(t)$ and the fraction of wealth invested in a riskless asset at time t is signified by $1-\theta(t)$. For any self-

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financing portfolio, the instantaneous change in the wealth return is then formulated as

$$\frac{dW(t)}{W(t)} = [1-\theta(t)]\frac{dB(t)}{B(t)} + \theta(t)\frac{dS(t)}{S(t)}$$

The wealth $W(t)$ of the investor can be expressed in the following stochastic process:

$$dW(t) = W(t)[r + \theta(t)(\mu - r)]dt + W(t)\theta(t)\sigma d\omega(t) \qquad (3)$$

The initial wealth of the investor is denoted by $W(0)$.

When markets are assumed complete, Harrison and Pliska (1981) showed that a unique state price density exits, $f(\zeta(T))$, which is log-normal. Let Z be the standardized value of $\ln\zeta(T)$

$$Z = \frac{\ln\zeta(T) - m}{v} \qquad (4)$$

where $m = \ln(\zeta(t)) - (r + \frac{1}{2}\| \kappa \|^2)(T-t)$ and $v = \| \kappa \| \sqrt{T-t}$ are the conditional mean and standard deviation of $\ln \zeta(T)$ implied by the risk-neutral pricing measure Q. $\kappa \equiv (\mu - r) / \sigma$ is the market price of risk. The density function of Z , the standardized value of $\ln\zeta(T)$, is given by

$$f(z) = \phi(z) \qquad (5)$$

where $\phi(\cdot)$ denotes the standard normal density function.

(B) Asset returns with fat-tailed distribution

Distribution with high tail probability compared to a normal distribution with the same mean and variance are called fat-tailed. Skewness and kurtosis measure the shape of a probability distribution. Skewness measures the degree of asymmetry, with symmetry implying zero skewness, and negative skewness indicates a relatively long left tail compared to the right. Tails of a distribution means that the regions are located far from center. Kurtosis is the extent to which probability in concentrated at the center, and especially the tails of the distributian. Because kurtosis is especially sensitive to tail-weight, high kurtosis is almost synonymous with fat tails. A fat-tailed distribution is more prone to extreme values, which are sometimes called outliers.

In this section, a riskless bond is presented and model of the stock index is developed that allows for negatively skewed and fat tails. In other words, stock index return skewness

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is typically negative and kurtosis for stock index return is positive. The Gram-Charlier expansion accommodates a virtually unlimited class of possible shapes, and yields an approximate density function for a standardized random variable. This study considers a Gram-Charlier expansion of the density function of the state price. The state price density process (or pricing kernel) has a critical role in deriving optimal trading strategies. Longstaff (1995), Corrado and Su (1996, 1997), and Jondeau and Rockinger (2001) applied a Gram-Charlier expansion for the underlying asset to develop risk-neutral density and derive an option pricing model.

Let Z be the standardized value of $\ln\zeta(T)$

$$Z = \frac{\ln\zeta(T) - m}{v}$$

Furthermore, let $f(Z;\ \beta,\ \delta)$ denote the risk-neutral density for Z implied by Q. For the lognormal density function, the Gram-Charlier expansion is given by

$$f(z;\beta,\delta) = (1+\beta(z^3-3z)+\delta(z^4-6z^2+3))\phi(z) \qquad (6)$$

where $\beta = \frac{\varsigma}{6}$, $\delta = \frac{\chi}{24}$ and ς , χ denote skewness and kurtosis of the density function, $\ln\ \zeta(T)$, respectively. Assuming the existence of one risky asset only, ς and χ also denote the skewness and kurtosis of the return on this risky asset. Under a normal specification, the skewness and kurtosis coefficients $\varsigma = 0$ and $\chi = 0$ respectively; by substituting them into $f(Z;\ \beta,\ \delta)$ generate the special case of a standard normal density $\phi(\cdot)$. When the stock index is developed that allows for negatively skewed and fat tails, then the state price skewness and kurtosis are positive (Jondeau & Rockinger, 2001).

C. Portfolio Optimization under VaR-RM

This section considers the dynamic investment problem of a VaR risk manager. The manager's objective is to maximize expected utility and maintain the VaR of terminal wealth at a prespecified level. The investor's utility function is assumed to belong to the class of constant relative risk aversion (CRRA) utility functions

$$U(W(T)) = \frac{W(T)^{1-\gamma}}{1-\gamma}, \qquad (7)$$

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and $\gamma > 0$ is the parameter of constant relative risk aversion. The optimal strategy that applies to the manager can be formulated as

$$\max_{W(T)} E[\frac{W(T)^{1-\gamma}}{1-\gamma}] \tag{8}$$

subject to

$$E[\zeta(T)W(T)] = \zeta(0)W(0),$$

$$P(W(T) \geq \underline{W}) \geq 1-\alpha.$$

The wealth process is assumed to be a self-financing process. In this case, the portfolio problem can be solved by the martingale solution technique proposed by Cox and Huang (1989, 1991). Following Cox and Huang (1989), the investor's problem is solved in two steps. First, the investor's optimal terminal wealth is determined, then the dynamic strategy that replicates it is derived. Basak and Shapiro (2001) showed that the solution to this program is

$$W(T)^{VaR} = \begin{cases} (y\zeta(T))^{-1/\gamma}, & if \quad \zeta(T) \leq \underline{\zeta} \\ \underline{W}, & if \quad \underline{\zeta} \leq \zeta(T) \leq \overline{\zeta} \\ (y\zeta(T))^{-1/\gamma}, & if \quad \overline{\zeta} \leq \zeta(T) \end{cases} \tag{9}$$

where $\underline{W} \equiv (y\underline{\zeta})^{-1/\gamma}$, $\overline{\zeta}$ is such that $P(\zeta(T) > \overline{\zeta}) \equiv \alpha$ and $\underline{\underline{W}} \equiv (y\overline{\zeta})^{-1/\gamma}$. Lagrange multiplier $y \geq 0$ solves $E(\zeta(T)\, W(T; y)) = \zeta(0)W(0)$.

The optimal wealth at each date t can be computed according to

$$\zeta(t)W(t)^{VaR} = E[\zeta(T)W(T)^{VaR}] \tag{10}$$

where E is the expected condition on the information available at time t. Using the martingale approach is in principle straightforward for solving the optimal wealth process described in the Appendix B.

To perform a detailed examination of the optimal behavior under the VaR-RM strategy, this study specialize the setting to Gram-Charlier expansion of the density function of state prices with a constant interest rate and market price of risk. Let β and δ denote the skewness and kurtosis of the density function, $\ln \zeta(T)$, respectively. Then

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Proposition 1:

The time t optimal wealth is given by

$$W^{VaR}(t)=\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}}\left[1+G_1(v')\right]$$

$$-\left\{\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}}\left[N\left(-d_1(\underline{\zeta})\right)+Q\left(-d_1(\underline{\zeta}),v'\right)\right]-\underline{W}e^{-r(T-t)}\left[N\left(-d_2(\underline{\zeta})\right)+Q\left(-d_2(\underline{\zeta}),v\right)\right]\right\}$$

$$+\left\{\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}}\left[N\left(-d_1(\overline{\zeta})\right)+Q\left(-d_1(\overline{\zeta}),v'\right)\right]-\underline{W}e^{-r(T-t)}\left[N\left(-d_2(\overline{\zeta})\right)+Q\left(-d_2(\overline{\zeta}),v\right)\right]\right\} \tag{11}$$

where $G_1(a)=\beta a^3+\delta a^4$

$$G_2(a,b)=\beta\left(a^2-3ab+3b^2-1\right)+\delta\left(-a^3+4a^2b-6ab^2+3a-4b+4b^3\right)$$

$$Q(a,b)=N(a)G_1(b)+\phi(a)G_2(a,b)$$

$$\Gamma(t)\equiv\left(\frac{1-\gamma}{\gamma}\right)\left(r+\frac{\|\kappa\|^2}{2}\right)(T-t)+\left(\frac{1-\gamma}{\gamma}\right)^2\frac{\|\kappa\|^2}{2}(T-t)$$

$$d_2(x)\equiv\frac{\ln\left(\frac{x}{\zeta(t)}\right)+\left(r-\frac{\|\kappa\|^2}{2}\right)(T-t)}{\|\kappa\|\sqrt{T-t}}$$

$$d_1(x)\equiv d_2(x)+\frac{1}{\gamma}\|\kappa\|\sqrt{T-t}$$

$$v'=\frac{\gamma-1}{\gamma}v\,.$$

Equation (11) represents time t optimal wealth that has three components: optimal wealth of a non-risk manager; cost of a put option on the benchmark-wealth, wealth of a non-risk manager, with strike price $\underline{W}$; and, the form of shorting a portfolio of a binary option. Consequently, the second and third terms attempt to keep time-T wealth at $\underline{W}$ in intermediate states. Notably, G and Q functions in equation (11) exhibit the impact of skewed and fat tails. The solutions when asset returns are normally distributed are special cases of the solutions presented when $\beta=0$ and $\delta=0$.

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Proposition 2:

The fraction of wealth invested in stock index is

$$\theta^{VaR}(t)=\frac{\left(\sigma(t)^{\mathrm{T}}\right)^{-1}\kappa^{\mathrm{T}}}{W(t)}\left(\frac{1}{\gamma}\right)\left\{W(t)-\underline{W}e^{-r(T-t)}\left[\mathrm{N}\left(-d_2(\underline{\zeta})\right)+Q\left(-d_2(\underline{\zeta}),v\right)-\mathrm{N}\left(-d_2(\overline{\zeta})\right)-Q\left(-d_2(\overline{\zeta}),v\right)\right]\right.$$

$$\left.-\frac{(\underline{\underline{W}}-\underline{W})e^{-r(T-t)}}{\|\kappa\|\sqrt{T-t}}\left[\phi\left(-d_2(\overline{\zeta})\right)\left(1+\varphi\left(\alpha(\overline{\zeta})\right)\right)\right]\right\},$$

Let $\theta^{VaR}(t)=q^{VaR}(t)\,\theta^{B}(t)$, where $\theta^{B}(t)$ is the benchmark value, the fraction of wealth invested in stock index of a non-risk manager, and $q^{VaR}(t)$ is the exposure to risky asset relative to the benchmark, and

$$\theta^{B}(t)=\frac{1}{\gamma}(\sigma(t)^{\mathrm{T}})^{-1}\kappa,$$

then

$$q^{VaR}(t)=1-\frac{\underline{W}e^{-r(T-t)}\left\{\left[\mathrm{N}\left(-d_2(\underline{\zeta})\right)-\mathrm{N}\left(-d_2(\overline{\zeta})\right)\right]+\left[Q\left(-d_2(\underline{\zeta}),v\right)-Q\left(-d_2(\overline{\zeta}),v\right)\right]\right\}}{W^{VaR}(t)}$$

$$+\frac{\gamma e^{-r(T-t)}\phi\left(d_2(\overline{\zeta})\right)(\underline{W}-\underline{\underline{W}})\left[1+\varphi\left(\alpha(\overline{\zeta})\right)\right]}{W^{VaR}(t)\|\kappa\|\sqrt{T-t}} \qquad (12)$$

where $\varphi(\alpha(x))=\beta\left[\alpha^3(x)-3\alpha(x)\right]+\delta\left[\alpha^4(x)-6\alpha^2(x)+3\right]$

$$\alpha(x)=\frac{\ln(x)-m}{v}=d_1(x)+v'=d_2(x)+v$$

The G, Q and φ functions of Eq. (12) exhibit the impact of skewed and fat tails. When $\beta=0$ and $\delta=0$, the G, Q and φ functions all equal 0, and the results in Eqs. (11) and (12) are the same as those obtained by Basak and Shapiro (2001), indicating that the solutions when asset returns are normally distributed are a special case of solutions presented when $\beta=0$ and $\delta=0$.

D. Numerical Illustrations

This section provides some numerical examples for analyzing how deviations from normality affect the end-of-horizon wealth and risk exposure of the VaR risk manager

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described in the previous section. The principal parameters for the financial market presented by Basak and Shapiro (2001) are applied. The parameters describing skewness and kurtosis by Corrado and Su (1996, 1997) are employed for numerical illustrations. Figures 1-5 show the impact of skewness and kurtosis on optimal terminal wealth and risk exposure of a VaR risk manager. Moreover, Figs. 1-4 plot the results obtained by Basak and Shapiro (2001) when $\beta = 0$ and $\delta = 0$.

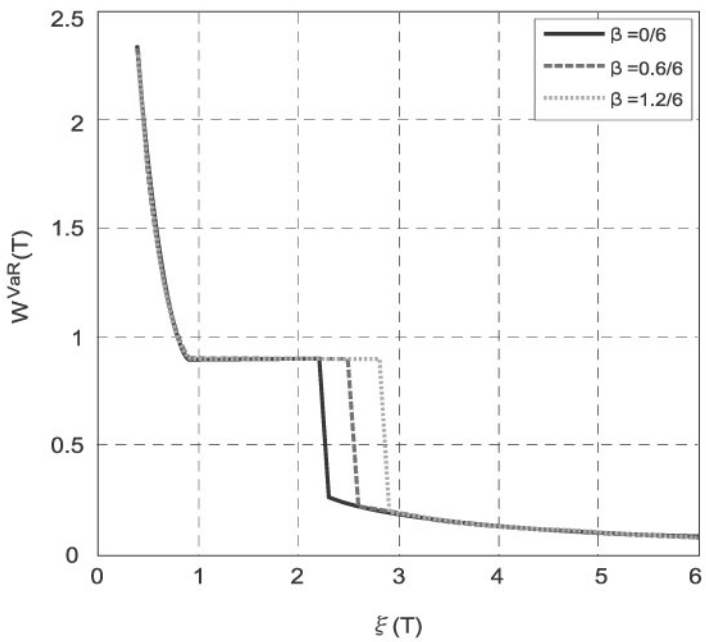


Figure 1 Optimal Terminal Wealth of a VaR Agent: Impact of Negative Skewness

Figure 1 presents the effect of negative skewness on optimal terminal wealth, $W(T)$, for a risk manager with $\delta = 0$, $\beta = 0$ (solid line), $\beta = 0.1$ (dashed line) and $\beta = 0.2$ (dotted line). The parameters utilized are $\gamma = 0.8$, $\alpha = 0.01$, $W_0 = 1.0$, $\underline{W} = 0.9$, $r = 0.05$, $\|\kappa\| = 0.4$, $T = 1$, $\zeta(0) = 1$. Then,

$\beta = 0$, $\delta = 0$: $\underline{\zeta} = 0.8625$, $\overline{\zeta} = 2.2267$, $\underline{\underline{W}} = 0.2750$;
$\beta = 0.1$, $\delta = 0$: $\underline{\zeta} = 0.8577$, $\overline{\zeta} = 2.5867$, $\underline{\underline{W}} = 0.2264$;
$\beta = 0.2$, $\delta = 0$: $\underline{\zeta} = 0.8502$, $\overline{\zeta} = 2.8334$, $\underline{\underline{W}} = 0.1999$.

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Figure 1 shows how negative skewness affects optimal terminal wealth. The agent behaves similarly to a benchmark agent, a non-risk manager, in both good states (low $\zeta(T)$) and bad states (high $\zeta(T)$) , and the manager insures against in intermediate states. Optimal terminal wealth is larger than $\underline{W}$ in good states, and optimal terminal wealth equals $\underline{W}$ in intermediate states. The probability in bad states equals α. α is set to 0.01 ; in other words, the probability of optimal terminal wealth being less than $\underline{W}$ is 0.01. However, optimal terminal wealth is far less than $\underline{W}$ in bad states and is less than $\underline{\underline{W}}$ in bad states. The impact of negative skewness can be ignored in both good and intermediated states. The agent will experience a great loss in bad states when confronted with negative skewness. Figure 2 presents the effect caused by excess kurtosis. The impact of excess kurtosis is identical to that of negative skewness.

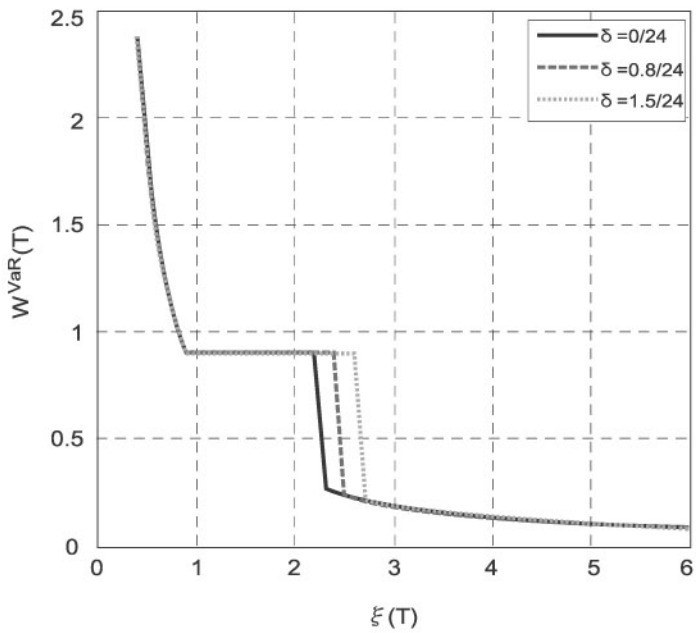


Figure 2 Optimal Terminal Wealth of a VaR Agent: Impact of Excess Kurtosis

Figure 2 plots the effect of excess kurtosis on optimal terminal wealth, $W(T)$, for a risk manager with $\beta = 0$, $\delta = 0$ (solid line), $\delta = 0.8/24$ (dashed line) and $\delta = 1.5/24$ (dotted line). The parameters used are $\gamma = 0.8$, $\alpha = 0.01$, $W_0 = 1.0$, $\underline{W} = 0.9$, $r = 0.05$, $||\kappa|| = 0.4$, $T = 1$, $\zeta(0) = 1$. Then,

$\beta = 0$, $\delta = 0$: $\underline{\zeta} = 0.8625$, $\overline{\zeta} = 2.2267$, $\underline{\underline{W}} = 0.2750$;

$\beta = 0$, $\delta = 0.8/24$: $\underline{\zeta} = 0.8679$, $\overline{\zeta} = 2.4344$, $\underline{\underline{W}} = 0.2479$;

$\beta = 0$, $\delta = 1.5/24$: $\underline{\zeta} = 0.8718$, $\overline{\zeta} = 2.6446$, $\underline{\underline{W}} = 0.2248$.

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Figure 3 illustrates how negative skewness and excess kurtosis affect optimal terminal wealth. The agent will have a substantial loss in bad states when confronted with negative skewness and excess kurtosis. When risky asset returns follow a normal distribution ($\beta = 0$, $\delta = 0$, solid line), the probability of optimal terminal wealth being below $\underline{\underline{W}}_1$ is 0.05 and $\underline{\underline{W}}_1$ is 0.4256. When risky asset return has negative skewness and excess kurtosis ($\beta = 0.2$, $\delta = 1.5 / 24$, dashed line), the probability of optimal terminal wealth being less than $\underline{\underline{W}}_2$ is 0.05 and $\underline{\underline{W}}_2$ is 0.3301. Moreover, when risky asset return has negative skewness and excess kurtosis, and the VaR manager allocates their portfolio based on the assumption of normal asset return, then the agent will experience an unexpected and great loss when the end-of-horizon state is bad. However, the VaR manager may mistakenly believe that the probability of achieving optimal terminal wealth of below 0.3301 is 0.0138. In fact, that probability is 0.05.

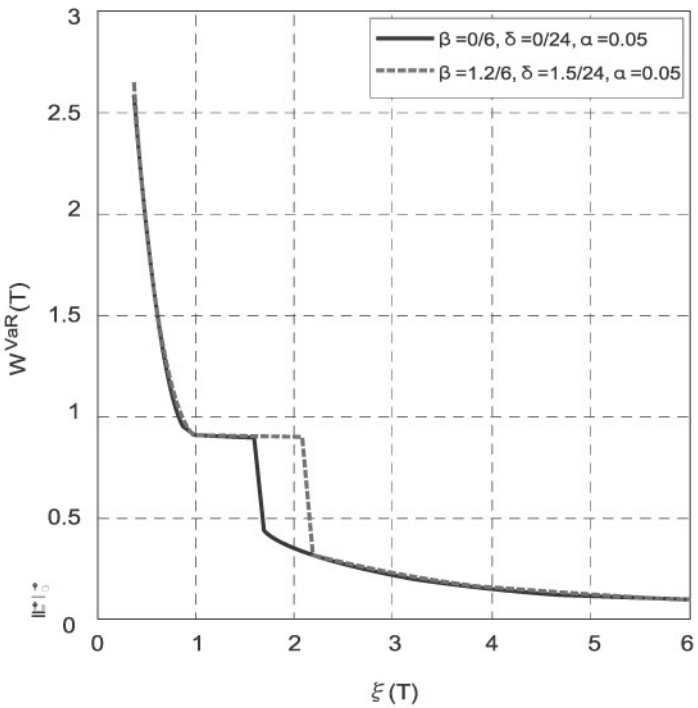


Figure 3 Optimal Terminal Wealth of a VaR Agent: Impact of Negative Skewness and Excess Kurtosis

Figure 3 plots the effects of negative skewness and excess kurtosis on optimal terminal wealth, $W(T)$ for a risk manager with $\beta = 0$, $\delta = 0$ (solid line), $\beta = 0.2$, $\delta = 1.5 / 24$ (dashed line). The parameters used are $\gamma = 0.8$, $\alpha = 0.05$, $W_0 = 1.0$, $\underline{W} = 0.9$, $r = 0.05$, $\|\kappa\| = 0.4$, $T = 1$, $\zeta(0) = 1$. Then,

$\beta = 0$, $\delta = 0$: $\underline{\zeta} = 0.9313$, $\overline{\zeta} = 1.6954$, $\underline{\underline{W}}_1 = 0.4256$;

$\beta = 0.2$, $\delta = 1.5 / 24$: $\underline{\zeta} = 0.9498$, $\overline{\zeta} = 2.1188$, $\underline{\underline{W}}_2 = 0.3301$.

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Figure 4 presents the influence of negative skewness and excess kurtosis on risk exposure. The risk manager invests a higher fraction of his wealth in the bond in the good states than a non-risk manager. As $\zeta(t)$ increases, the VaR agent begins to increase his equity exposure and invests a higher proportion of his wealth in the stock index. When $\zeta(t)$ becomes sufficiently high to deter the agent from taking further risks, the agent returns to his benchmark strategy. Asset allocation attempts to replicate a portfolio of binary options and insuring the intermediate states region. When a risk manager confronts negative skewness and excess kurtosis, the agent invests reduced funds in the stock market in intermediate states and increase their investment in the stock market in bad states.

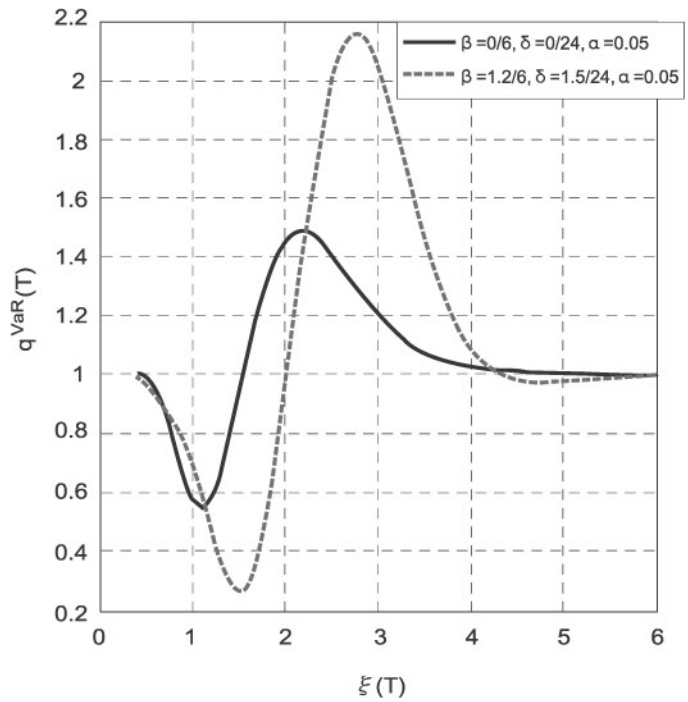


Figure 4 Risk Exposure of a VaR Agent: Impact of Negative Skewness and Excess Kurtosis

Figure 4 presents the effect of negative skewness and excess kurtosis on exposure to risky assets relative to the benchmark, $q(t)$, for a risk manager with , $\beta = 0$, $\delta = 0$ and $\alpha = 0.05$ (solid line), $\beta = 0.2$, $\delta = 1.5 / 24$ and $\alpha = 0.05$ (dashed line). The parameters used are $\gamma = 0.8$, $W_0 = 1.0$, $\underline{W} = 0.9$, $r = 0.05$, $\|\kappa\| = 0.4$, $T = 1$, $t = 0.5$, $\zeta(0) = 1$.

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Importantly, a risk manager must forecast the loss precisely and to estimate the probability that the end-of-horizon state is bad. When risky asset returns have negative skewness and excess kurtosis, the agent will experience an unexpected and large loss when the end-of-horizon state is a bad state (Fig. 3). A risk manager cannot decrease the loss in a bad state, but can decrease the value of α, the probability that a loss exceeds VaR, and moreover, the agent will simultaneously have reduced terminal wealth in both the good and bad states. Figure 5 illustrates this result.

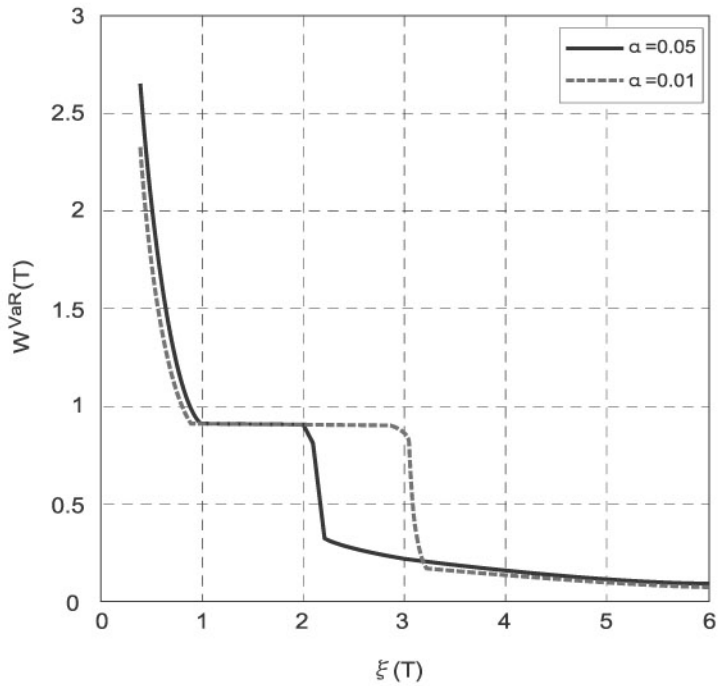


Figure 5 Optimal Terminal Wealth of a VaR Agent: Impact of α

Figure 5 shows the effect of α on optimal terminal wealth, $W(T)$, for a risk manager with $\beta = 0.2$, $\delta = 1.5/24$ and $\alpha = 0.05$ (solid line), $\beta = 0.2$, $\delta = 1.5/24$ and $\alpha = 0.01$ (dashed line). Parameters used are $\gamma = 0.8$, $W_0 = 1.0$, $\underline{W} = 0.9$, $r = 0.05$, $\|\kappa\| = 0.4$, $T = 1$, $\zeta(0) = 1$.

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E. Conclusions

This study investigated the effect of extreme returns or fat tails on the solution of the asset allocation problem for risk managers. This study provided solutions to the intertemporal investment strategy by an agent in a complete market setting in which asset returns are negatively skewed and have excess kurtosis. Therefore, the Gram-Charlier expansion was utilized to accommodate negatively skewed and excess kurtosis.

How do extreme returns or fat tails affect optimal asset allocation and end-of-horizon wealth? First, when risky asset return has negative skewness and excess kurtosis, and the VaR manager allocates their portfolio under the assumption of normal asset return, then the agent will experience an unexpected and great loss when the end-of-horizon state is a bad state. Second, the agent invests reduced funds in the stock market during intermediate states and increased amounts in the stock market during bad states when confronted with negative skewness and excess kurtosis. Third, a risk manager cannot decrease losses in bad states, but can decrease the value of α, the probability that a loss exceeds VaR, and the agent will suffer from reduced terminal wealth in both the good and bad states.

The impact of skewness and kurtosis is large and should not be neglected. Analytical results suggest that the total impact of leptokurtic asset returns depends on the shape of asset returns and correctly measuring leptokurtic asset returns is very important for optimal asset choice made by risk managers.

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Appendix

A. This appendix shows the detailed explanation of section B.

$\kappa \equiv (\mu - r) / \sigma$,

If $E[\exp(\frac{1}{2}\int_0^T \kappa^2 ds)] < \infty$, then the Novikov condition holds and the state price density of the economy $\zeta(t)$ exists, as follows:

$$\zeta(t) = \exp\left\{-\int_0^t r ds - \int_0^t \kappa d\omega(s) - \frac{1}{2}\int_0^t \kappa^2 ds\right\}$$

$$= \exp\left\{-rt - \kappa\omega(t) - \frac{1}{2}\kappa^2 t\right\}.$$

and $$\frac{\zeta(T)}{\zeta(t)} = \exp\left\{-r(T-t) - \kappa[\omega(T) - \omega(t)] - \frac{1}{2}\kappa^2(T-t)\right\},$$

$$\ln\zeta(T) = \left\{\ln\zeta(t) - (r + \frac{1}{2}\kappa^2)(T-t) - \kappa[\omega(T-t)]\right\}.$$

Let $m = \ln(\zeta(t)) - (r + \frac{1}{2}\|\kappa\|^2)(T-t)$, $v = \|\kappa\|\sqrt{T-t}$ and

$$Z = \frac{\ln\zeta(T) - m}{v} = \frac{-\kappa\omega(T-t)}{v}.$$

When risky asset returns follow a normal distribution (Section B (A)), then $\frac{-\kappa\omega(T-t)}{v}$ also follows a standard normal distribution. When the stock index is developed that allows for negatively skewed and fat tails (Section B (B)), then stock index return skewness is typically negative and kurtosis is positive. $\frac{\kappa\omega(T-t)}{v}$ follows Gram-Charlier expansion of the density function. $\frac{\kappa\omega(T-t)}{v}$ has negative skewness and positive kurtosis. Since skewness denotes the third moment and kurtosis denotes the fourth moment, skewness for $\frac{-\kappa\omega(T-t)}{v}$ is positive and kurtosis for $\frac{-\kappa\omega(T-t)}{v}$ is still positive.

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B. Proof of equation (11)

Let $f(Z;\beta,\delta)$ denote the risk-neutral density for Z implied by Q. For the lognormal density function, the Gram-Charlier expansion is given by

$$f(z;\beta,\delta)=(1+\beta(z^3-3z)+\delta(z^4-6z^2+3))\phi(z)$$

where $Z=\dfrac{\ln\zeta(T)-m}{v}$, $m=\ln(\zeta(t))-(r+\dfrac{1}{2}\|\kappa\|^2)(T-t)$, $v=\|\kappa\|\sqrt{T-t}$.

From (9),

$$W^{VaR}(T)=\begin{cases}(y\zeta(\mathrm{T}))^{\frac{1}{\gamma}}, & if \quad \zeta(T)\le\underline{\zeta}\\ \underline{W}, & if \quad \underline{\zeta}\le\zeta(T)\le\overline{\zeta}\\ (y\zeta(\mathrm{T}))^{\frac{1}{\gamma}}, & if \quad \overline{\zeta}\le\zeta(T)\end{cases}$$

The optimal wealth at each date t

$$W^{VaR}(t)=E\left[\frac{\zeta(\mathrm{T})}{\zeta(t)}W(\mathrm{T})\middle|F_t\right]$$

$$=\int_0^{\underline{\zeta}}\zeta(\mathrm{T})\frac{(y\zeta(\mathrm{T}))^{-\frac{1}{\gamma}}}{\zeta(t)}f(\zeta(\mathrm{T}))d\zeta(\mathrm{T})+\int_{\underline{\zeta}}^{\overline{\zeta}}\zeta(\mathrm{T})\frac{\underline{W}}{\zeta(t)}f(\zeta(\mathrm{T}))d\zeta(\mathrm{T})$$

$$+\int_{\overline{\zeta}}^{\infty}\zeta(\mathrm{T})\frac{(y\zeta(\mathrm{T}))^{-\frac{1}{\gamma}}}{\zeta(t)}f(\zeta(\mathrm{T}))d\zeta(\mathrm{T})$$

$$=\frac{1}{\zeta(\mathrm{t})\mathrm{y}^{\frac{1}{\gamma}}}\int_{-\infty}^{k_1}e^{\frac{(zv+m)^{\gamma-1}}{\gamma}}\left[1+\beta(z^3-3z)+\delta(z^4-6z^2+3)\right]\phi(z)dz$$

$$+\frac{\underline{W}}{\zeta(t)}\int_{k_1}^{k_2}e^{(zv+m)}\left[1+\beta(z^3-3z)+\delta(z^4-6z^2+3)\right]\phi(z)dz$$

$$+\frac{1}{\zeta(\mathrm{t})\mathrm{y}^{\frac{1}{\gamma}}}\int_{k_2}^{\infty}e^{\frac{(zv+m)^{\gamma-1}}{\gamma}}\left[1+\beta(z^3-3z)+\delta(z^4-6z^2+3)\right]\phi(z)dz$$

where $k_1=\dfrac{\ln(\underline{\zeta})-m}{v}$, $k_2=\dfrac{\ln(\overline{\zeta})-m}{v}$

$$=(1)+(2)+(3)$$

$$=\frac{e^{\Gamma(t)}}{(y\zeta(t))^{1/\gamma}}\left[1+G_1(v')\right]-\frac{e^{\Gamma(t)}}{(y\zeta(t))^{1/\gamma}}\left[N\left(-d_1(\underline{\zeta})\right)+Q\left(-d_1(\underline{\zeta}),v'\right)\right]$$

$$-\underline{W}e^{-r(T-t)}\left[-N\left(-d_2(\underline{\zeta})\right)-Q\left(-d_2(\underline{\zeta}),v\right)+N\left(-d_2(\overline{\zeta})\right)+Q\left(-d_2(\overline{\zeta}),v\right)\right]$$

$$+\frac{e^{\Gamma(t)}}{(y\zeta(t))^{1/\gamma}}\left[N\left(-d_1(\overline{\zeta})\right)+Q\left(-d_1(\overline{\zeta}),v'\right)\right]$$

$$=\frac{e^{\Gamma(t)}}{(y\zeta(t))^{1/\gamma}}\left[1+G_1(v')\right]$$

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$$-\left\{\frac{e^{\Gamma(t)}}{(y\zeta(t))^{1/\gamma}}\left[N\left(-d_1(\underline{\zeta})\right)+Q\left(-d_1(\underline{\zeta}),v'\right)\right]-\underline{W}e^{-r(T-t)}\left[N\left(-d_2(\underline{\zeta})\right)+Q\left(-d_2(\underline{\zeta}),v\right)\right]\right\}$$

$$+\left\{\frac{e^{\Gamma(t)}}{(y\zeta(t))^{1/\gamma}}\left[N\left(-d_1(\overline{\zeta})\right)+Q\left(-d_1(\overline{\zeta}),v'\right)\right]-\underline{W}e^{-r(T-t)}\left[N\left(-d_2(\overline{\zeta})\right)+Q\left(-d_2(\overline{\zeta}),v\right)\right]\right\}$$

where $G_1(a)=\beta a^3+\delta a^4$

$$G_2(a,b)=\beta\left(a^2-3ab+3b^2-1\right)+\delta\left(-a^3+4a^2b-6ab^2+3a-4b+4b^3\right)$$

$$Q(a,b)=N(a)G_1(a)+\phi(a)G_2(a,b)$$

$$\Gamma(t)\equiv\left(\frac{1-\gamma}{\gamma}\right)\left(r+\frac{\|\kappa\|^2}{2}\right)(T-t)+\left(\frac{1-\gamma}{\gamma}\right)^2\frac{\|\kappa\|^2}{2}(T-t)$$

$$d_2(x)\equiv\frac{\ln\left(\frac{x}{\zeta(t)}\right)+\left(r-\frac{\|\kappa\|^2}{2}\right)(T-t)}{\|\kappa\|\sqrt{T-t}}$$

$$d_1(x)\equiv d_2(x)+\frac{1}{\gamma}\|\kappa\|\sqrt{T-t}$$

$$v'=\frac{\gamma-1}{\gamma}v$$

$$(1)=\frac{e^{m\frac{\gamma-1}{\gamma}}}{\zeta(t)y^{\frac{1}{\gamma}}}\left\{\int_{-\infty}^{k_1}e^{zv\frac{\gamma-1}{\gamma}}\phi(z)dz+\beta\left[\int_{-\infty}^{k_1}e^{zv\frac{\gamma-1}{\gamma}}z^3\phi(z)dz-3\int_{-\infty}^{k_1}e^{zv\frac{\gamma-1}{\gamma}}z\phi(z)dz\right]\right.$$

$$\left.+\delta\left[\int_{-\infty}^{k_1}e^{zv\frac{\gamma-1}{\gamma}}z^4\phi(z)dz-6\int_{-\infty}^{k_1}e^{zv\frac{\gamma-1}{\gamma}}z^2\phi(z)dz+3\int_{-\infty}^{k_1}e^{zv\frac{\gamma-1}{\gamma}}\phi(z)dz\right]\right\}$$

$$=\frac{e^{m\frac{\gamma-1}{\gamma}+\frac{v'^2}{2}}}{\zeta(t)y^{\frac{1}{\gamma}}}\left\{N(d_1)\right.$$

$$+\beta\left[-(d_1^2+3v'd_1+3v'^2+2)\phi(d_1)+(3v'+v'^3)N(d_1)-3\left(-\phi(d_1)+v'N(d_1)\right)\right]$$

$$+\delta\left[-(d_1^3+3d_1+4v'd_1^2+8v'+6v'^2d_1+4v'^3)\phi(d_1)+(3+6v'^2+v'^4)N(d_1)\right.$$

$$\left.\left.-6\left(-(d_1+2v')\phi(d_1)+(1+v'^2)N(d_1)\right)+3N(d_1)\right]\right\},\qquad\text{(by(4))}$$

where $d_1=d_1(\underline{\zeta})=k_1-v'$

$$=\frac{e^{\Gamma(t)}\zeta(t)^{1-\frac{1}{\gamma}}}{\zeta(t)y^{\frac{1}{\gamma}}}\left\{-\phi(d_1)\left[\beta(d_1^2+3d_1v'+3v'^2-1)+\delta(d_1^3-3d_1+4d_1^2v'-4v'+6d_1v'^2+4v'^3)\right]\right.$$

$$\left.+N(d_1)\left[1+\beta v'^3+\delta v'^4\right]\right\}$$

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$$= \frac{e^{\Gamma(t)}}{(\zeta(t)y)^{\frac{1}{\gamma}}}\left\{ -\phi(-d_1)G_2(-d_1,v') + [1-N(-d_1)][(1+G_1(v')]\right\}$$

$$= \frac{e^{\Gamma(t)}}{(\zeta(t)y)^{\frac{1}{\gamma}}}[1+G_1(v')] - \frac{e^{\Gamma(t)}}{(\zeta(t)y)^{\frac{1}{\gamma}}}\left\{\phi(-d_1(\underline{\zeta})G_2(-d_1(\underline{\zeta}),v') + N\left(-d_1(\underline{\zeta})\right)[1+G_1(v')]\right\}$$

$$= \frac{e^{\Gamma(t)}}{(\zeta(t)y)^{\frac{1}{\gamma}}}[1+G_1(v')] - \frac{e^{\Gamma(t)}}{(\zeta(t)y)^{\frac{1}{\gamma}}}\left[N\left(-d_1(\underline{\zeta})\right) + Q\left(-d_1(\underline{\zeta}),v'\right)\right]$$

$$(2) = \frac{\underline{W}e^{m}}{\zeta(t)}\left\{ \int_{k_1}^{k_2} e^{zv}\phi(z)dz + \beta\left[\int_{k_1}^{k_2} e^{zv} z^3\phi(z)dz - 3\int_{k_1}^{k_2} e^{zv} z\phi(z)dz \right] \right.$$
$$\left. + \delta\left[\int_{k_1}^{k_2} e^{zv} z^4\phi(z)dz - 6\int_{k_1}^{k_2} e^{zv} z^2\phi(z)dz + 3\int_{k_1}^{k_2} e^{zv}\phi(z)dz \right]\right\}$$

$$= \frac{\underline{W}e^{m+\frac{v^2}{2}}}{\zeta(t)}\left\{[N(d_4)-N(d_3)]\right.$$
$$+\beta\left[-(d_4^2+2+3d_4v+3v^2)\phi(d_4) + (d_3^2+2+3d_3v+3v^2)\phi(d_3) + 3(v+v^3)\left(N(d_4)-N(d_3)\right)\right]$$
$$+\delta\left[-(d_4^3+3d_4+4d_4^2v+8v+6d_4v^2+4v^3)\phi(d_4)\right.$$
$$+(d_3+3d_3+4d_3^2v+8v+6d_3v^2+4v^3)\phi(d_3) + (3+6v^2+v^4)\left(N(d_4)-N(d_3)\right)$$
$$+6(d_4+2v)\phi(d_4) - 6(d_3+2v)\phi(d_3) - 6(1+v^2)\left(N(d_4)-N(d_3)\right)$$
$$\left.\left.+3\left(N(d_4)-N(d_3)\right)\right]\right\}, \qquad \text{(by(4))}$$

Where $\quad d_3 = d_2(\underline{\zeta}) = k_1 - v; \qquad d_4 = d_2(\overline{\zeta}) = k_2 - v$

$$= \frac{\underline{W}e^{m+\frac{v^2}{2}}}{\zeta(t)}\left\{-\phi(d_4)\left[\beta(d_4^2+3d_4v+3v^2-1) + \delta(d_4^3+4d_4^2v+6d_4v^2-3d_4-4v+4v^3)\right]\right.$$
$$+\phi(d_3)\left[\beta(d_3^2+3d_3v+3v^2-1) + \delta(d_3^3+4d_3^2v+6d_3v^2-3d_3-4v+4v^3)\right]$$
$$\left.+(1+\beta v^3+\delta v^4)[N(d_4)-N(d_3)]\right\}$$

$$= \frac{\underline{W}e^{m+\frac{v^2}{2}}}{\zeta(t)}\left\{[1-N(-d_4)][(1+G_1(v)] - \phi(-d_4)G_2(-d_4,v) - [1-N(-d_3)][1+G_1(v)] + \phi(-d_3)G_2(-d_3,v)\right\}$$

$$= \frac{\underline{W}e^{-r(T-t)}\zeta(t)}{\zeta(t)}\left\{N(-d_2(\underline{\zeta})[1+G_1(v)] + \phi(-d_2(\underline{\zeta}))G_2(-d_2(\underline{\zeta}),v)\right.$$
$$\left.-N(-d_2(\overline{\zeta})[1+G_1(v)] - \phi(-d_2(\overline{\zeta}))G_2(-d_2(\overline{\zeta}),v)\right\}$$

$$= \underline{W}e^{-r(T-t)}\left\{\left[N(-d_2(\underline{\zeta})) + Q(-d_2(\underline{\zeta}),v)\right] - \left[N(-d_2(\overline{\zeta})) + Q(-d_2(\overline{\zeta}),v)\right]\right\}$$

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$$(3) = \frac{e^{m\frac{\gamma-1}{\gamma}}}{\zeta(t)\,y^{\frac{1}{\gamma}}}\left\{\int_{k_2}^{\infty} e^{zv\frac{\gamma-1}{\gamma}}\phi(z)dz + \beta\left[\int_{k_2}^{\infty} e^{zv\frac{\gamma-1}{\gamma}} z^3\phi(z)dz - 3\int_{k_2}^{\infty} e^{zv\frac{\gamma-1}{\gamma}} z\phi(z)dz\right]\right.$$

$$\left. + \delta\left[\int_{k_2}^{\infty} e^{zv\frac{\gamma-1}{\gamma}} z^4\phi(z)dz - 6\int_{k_2}^{\infty} e^{zv\frac{\gamma-1}{\gamma}} z^2\phi(z)dz + 3\int_{k_2}^{\infty} e^{zv\frac{\gamma-1}{\gamma}}\phi(z)dz\right]\right\}$$

$$= \frac{e^{m\frac{\gamma-1}{\gamma}+\frac{v'^2}{2}}}{\zeta(t)\,y^{\frac{1}{\gamma}}}\left\{\phi(d_s)\left[\beta(d_5^2+3d_5v'+3v'^2-1)+\delta(d_5^3+4d_5^2v'+6d_5v'^2-3d_5-4v'+4v'^3)\right]\right.$$

$$\left. + \left[1-N(d_5)\right](1+\beta v'^3+\delta v'^4)\right\}, \qquad \text{(by(4))}$$

where $d_5 = d_1(\overline{\zeta}) = k_2 - v'$

$$= \frac{e^{\Gamma(t)}\zeta(t)^{1-\frac{1}{\gamma}}}{\zeta(t)\,y^{\frac{1}{\gamma}}}\left\{\phi(-d_5)G_2(-d_5,v') + N(-d_5)\left[1+G_1(v')\right]\right\}$$

$$= \frac{e^{\Gamma(t)}}{(\zeta(t)y)^{\frac{1}{\gamma}}}\left[\,N(-d_5)+Q(-d_5,v')\,\right]$$

$$= \frac{e^{\Gamma(t)}}{(\zeta(t)y)^{\frac{1}{\gamma}}}\left[N\left(-d_1(\overline{\zeta})\right)+Q\left(-d_1(\overline{\zeta}),v'\right)\right]$$

(4)

a) $\int_{-\infty}^{k_1} e^{zv\frac{\gamma-1}{\gamma}}\phi(z)dz \qquad (Let\ \ t = z - v\frac{\gamma-1}{\gamma} = z - v')$

$= e^{\frac{v^2(\frac{\gamma-1}{\gamma})^2}{2}}\int_{-\infty}^{k_1 - v\frac{\gamma-1}{\gamma}}\phi(t)dt \qquad (Let\ \ d_1 = k_1 - v\frac{\gamma-1}{\gamma} = k_1 - v')$

$= e^{\frac{v'^2}{2}}N(d_1)$

b) $\int_{-\infty}^{k_1} e^{zv\frac{\gamma-1}{\gamma}} z\phi(z)dz \qquad (Let\ \ t = z - v\frac{\gamma-1}{\gamma} = z - v')$

$= e^{\frac{v'^2}{2}}\int_{-\infty}^{d_1}[t+v']\phi(t)dt \qquad (Let\ \ d_1 = k_1 - v\frac{\gamma-1}{\gamma} = k_1 - v')$

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$$= e^{\frac{v'^2}{2}}\left[\int_{-\infty}^{d_1} t\phi(t)dt + v'\int_{-\infty}^{d_1}\phi(t)dt\right]$$

$$= e^{\frac{v'^2}{2}}\left[-\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\Big|_{-\infty}^{d_1} + v'N(d_1)\right]$$

$$= e^{\frac{v'^2}{2}}\left[-\phi(d_1) + v'N(d_1)\right]$$

where $\int t\phi(t)dt = -\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}} + c$

c) $\int_{-\infty}^{k_1} e^{zv\frac{\gamma-1}{\gamma}} z^2\phi(z)dz$ $\quad (Let\ \ t = z - v\frac{\gamma-1}{\gamma} = z - v')$

$$= e^{\frac{v'^2}{2}}\left[\int_{-\infty}^{d_1} t^2\phi(t)dt + 2v'\int_{-\infty}^{d_1} t\phi(t)dt + v'^2\int_{-\infty}^{d_1}\phi(t)dt\right]$$

$(Let\ \ d_1 = k_1 - v\frac{\gamma-1}{\gamma} = k_1 - v')$

$$= e^{\frac{v'^2}{2}}\left\{-d_1\phi(d_1) + N(d_1) + 2v'\left[-\phi(d_1)\right] + v'^2 N(d_1)\right\}$$

$$= e^{\frac{v'^2}{2}}\left\{-(d_1 + 2v')\phi(d_1) + (1 + v'^2)N(d_1)\right\}$$

where $\int t^2\phi(t)dt = -t\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}} + \int\phi(t)dt$

d) $\int_{-\infty}^{k_1} e^{zv\frac{\gamma-1}{\gamma}} z^3\phi(z)dz$ $\quad (Let\ \ t = z - v\frac{\gamma-1}{\gamma} = z - v')$

$$= e^{\frac{v'^2}{2}}\left\{\int_{-\infty}^{d_1} t^3\phi(t)dt + 3v'\int_{-\infty}^{d_1} t^2\phi dt + 3v'^2\int_{-\infty}^{d_1} t\phi(t)dt + v'^3\int_{-\infty}^{d_1}\phi(t)dt\right\}$$

$$= e^{\frac{v'^2}{2}}\left\{-(d_1^2 + 2)\phi(d_1) + 3v'\left[-d_1\phi(d_1) + N(d_1)\right] + 3v'^2\left[-\phi(d_1)\right] + v'^3 N(d_1)\right\}$$

$$= e^{\frac{v'^2}{2}}\left\{-\left[d_1^2 + 3v'd_1 + 3v'^2 + 2\right]\phi(d_1) + \left[3v' + v'^3\right]N(d_1)\right\}$$

where $\int t^3\phi(t)dt = -t^2\phi(t) - 2\phi(t) + c$

e) $\int_{-\infty}^{k_1} e^{zv\frac{\gamma-1}{\gamma}} z^4\phi(z)dz$ $\quad (Let\ \ t = z - v\frac{\gamma-1}{\gamma} = z - v')$

$$= e^{\frac{v'^2}{2}}\left\{\int_{-\infty}^{d_1} t^4\phi(t)dt + 4v'\int_{-\infty}^{d_1} t^3\phi(t)dt + 6v'^2\int_{-\infty}^{d_1} t^2\phi(t)dt + 4v'^3\int_{-\infty}^{d_1} t\phi(t)dt + v'^4\int_{-\infty}^{d_1}\phi(t)dt\right\}$$

$$= e^{\frac{v'^2}{2}}\left\{-(d_1^3 + 3d_1)\phi(d_1) + 3N(d_1) + 4v'\left[-(d_1^2 + 2)\phi(d_1)\right]\right.$$
$$\left. + 6v'^2\left[-d_1\phi(d_1) + N(d_1)\right] + 4v'^3\left[-\phi(d_1)\right] + v'^4 N(d_1)\right\}$$

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$$= e^{\frac{v'^2}{2}}\left\{-\left[d_1^3+3d_1+4d_1^2v'+8v'+6d_1v'^2+4v'^3\right]\phi(d_1)+\left[3+6v'^2+v'^4\right]N(d_1)\right\}$$

where $\int t^4\phi(t)dt = -t^3\phi(t)-3t\phi(t)+3\int\phi(t)dt$

Proof of equation (12)

By Ito Lemma, we have

$$dW(t) = \frac{\partial W(t)}{\partial t}dt+\frac{\partial W(t)}{\partial \zeta(t)}d\zeta(t) \qquad (\because d\zeta(t) = -\zeta(t)\left[r(t)dt+\kappa^{\mathrm{T}}d\omega(t)\right])$$

$$= G(t,\zeta(t))dt-\frac{\partial W(t)}{\partial \zeta(t)}\zeta(t)\kappa^{\mathrm{T}}d\omega(t) \quad \text{———}$$

$$\because dW(t) = W(t)\left[r(t)+\theta(t)^{\mathrm{T}}\left(\mu(t)-r(t)\bar{1}\right)\right]dt+W(t)\theta(t)^{\mathrm{T}}\sigma(t)d\omega(t) \quad \text{——}$$

$$= \quad , \therefore W(t)\theta(t)^{\mathrm{T}}\sigma(t) = -\frac{\partial W(t)}{\partial \zeta(t)}\zeta(t)\kappa^{\mathrm{T}}$$

$$\therefore \theta(t) = -\frac{\left(\sigma(t)^{\mathrm{T}}\right)^{-1}\kappa^{\mathrm{T}}}{W(t)}\left(\frac{\partial W(t)}{\partial \zeta(t)}\zeta(t)\right)$$

$$\Rightarrow \frac{\partial W(t)}{\partial \zeta(t)} = -\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}}\left[1+G_1(v')\right]\left(\frac{1}{\gamma}\right)\zeta(t)^{-1}$$

$$-\left\{-\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}}\left[\mathrm{N}\left(-d_1(\underline{\zeta})\right)+Q\left(-d_1(\underline{\zeta}),v'\right)\right]\left(\frac{1}{\gamma}\right)\zeta(t)^{-1}\right.$$

$$+\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}}\cdot\frac{\phi\left(-d_1(\underline{\zeta})\right)}{\zeta(t)\,\|\,\kappa\,\|\sqrt{T-t}}\cdot\left[1+\varphi\left(\alpha(\underline{\zeta})\right)\right]$$

$$\left.-\underline{W}e^{-r(T-t)}\frac{\phi\left(-d_2(\underline{\zeta})\right)}{\zeta(t)\,\|\,\kappa\,\|\sqrt{\mathrm{T}-t}}\cdot\left[1+\varphi\left(\alpha(\underline{\zeta})\right)\right]\right\}$$

$$+\left\{-\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}}\left[\mathrm{N}\left(-d_1(\overline{\zeta})\right)+Q\left(-d_1(\overline{\zeta}),v'\right)\right]\left(\frac{1}{\gamma}\right)\zeta(t)^{-1}\right.$$

$$+\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}}\cdot\frac{\phi\left(-d_1(\overline{\zeta})\right)}{\zeta(t)\,\|\,\kappa\,\|\sqrt{T-t}}\cdot\left[1+\varphi\left(\alpha(\overline{\zeta})\right)\right]$$

$$\left.-\underline{W}e^{-r(T-t)}\frac{\phi\left(-d_2(\overline{\zeta})\right)}{\zeta(t)\,\|\,\kappa\,\|\sqrt{\mathrm{T}-t}}\cdot\left[1+\varphi\left(\alpha(\overline{\zeta})\right)\right]\right\}$$

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where

$$\frac{\partial}{\partial\zeta(t)}\left[N(-d_1(\underline{\zeta})+Q(-d_1(\underline{\zeta}),\ v')\right] \qquad \left[\because Q(a,b)=N(a)G_1(b)+\phi(a)G_2(a,b)\right]$$

$$=\frac{\partial N\left(-d_1(\underline{\zeta})\right)}{\partial\zeta(t)}+\frac{\partial N\left(-d_1(\underline{\zeta})\right)}{\partial\zeta(t)}G_1(v')+\frac{\partial N\left(-d_1(\underline{\zeta})\right)}{\partial\zeta(t)}G_2(-d_1(\underline{\zeta}),v')+\phi\left(-d_1(\underline{\zeta})\right)\frac{\partial G_2\left(-d_1(\underline{\zeta}),v'\right)}{\partial\zeta(t)}$$

$$=\phi\left(-d_1(\underline{\zeta})\right)\frac{1}{\zeta(t)}\frac{1}{\|\kappa\|\sqrt{T-t}}\left[1+G_1(v')\right]+\phi\left(-d_1(\underline{\zeta})\right)d_1(\underline{\zeta})\frac{1}{\zeta(t)}\frac{1}{\|\kappa\|\sqrt{\tau-t}}G_2\left(-d_1(\underline{\zeta}),v'\right)$$

$$-\phi\left(-d_1(\underline{\zeta})\right)\frac{1}{\zeta(t)}\frac{1}{\|\kappa\|\sqrt{T-t}}\left[\beta(2d_1+3v')+\delta(3d_1^2+8d_1v'+6v'^2-3)\right]$$

$$=\frac{\phi\left(-d_1(\underline{\zeta})\right)}{\zeta(t)\|\kappa\|\sqrt{T-t}}\left\{1+G_1(v')+d_1(\underline{\zeta})G_2\left(-d_1(\underline{\zeta}),\sigma'\right)-\left[\beta(2d_1+3v')+\delta(3d_1^2+8d_1v'+6v'^2-3)\right]\right\}$$

$$=\frac{\phi\left(-d_1(\underline{\zeta})\right)}{\zeta(t)\|\kappa\|\sqrt{T-t}}\left\{1+\beta\left[\alpha^3(\underline{\zeta})-3\alpha(\underline{\zeta})\right]+\delta\left[\alpha^4(\underline{\zeta})-6\alpha^2(\underline{\zeta})+3\right]\right\}$$

$$=\frac{\phi\left(-d_1(\underline{\zeta})\right)}{\zeta(t)\|\kappa\|\sqrt{T-t}}\left[1+\varphi\left(\alpha(\underline{\zeta})\right)\right]$$

$$\because \frac{\partial N\left(-d_1(\underline{\zeta})\right)}{\partial\zeta(t)}=\phi\left(-d_1(\underline{\zeta})\right)\frac{\partial\left(-d_1(\underline{\zeta})\right)}{\partial\zeta(t)}=\phi\left(-d_1(\underline{\zeta})\right)\frac{1}{\zeta(t)}\frac{1}{\|\kappa\|\sqrt{T-t}}$$

$$\because \frac{\partial G_2\left(-d_1(\underline{\zeta})\right),v')}{\partial\zeta(t)}$$

$$=\beta\left[2d_1\frac{\partial d_1}{\partial\zeta(t)}+3v'\frac{\partial d_1}{\partial\zeta(t)}\right]+\delta\left[3d_1^2\frac{\partial d_1}{\partial\zeta(t)}+8d_1v'\frac{\partial d_1}{\partial\zeta(t)}+6v'^2\frac{\partial d_1}{\partial\zeta(t)}-3\frac{\partial d_1}{\partial\zeta(t)}\right]$$

$$=\frac{\partial d_1}{\partial\zeta(t)}\left[\beta(2d_1+3v')+\delta(3d_1^2+8d_1v'+6v'^2-3)\right]$$

$$=\frac{-1}{\zeta(t)}\frac{1}{\|\kappa\|\sqrt{T-t}}\left[\beta(2d_1+3v')+\delta(3d_1^2+8d_1v'+6v'^2-3)\right]$$

$$\Rightarrow\theta^{VaR}(t)$$

$$=\frac{\left(\sigma(t)^{\mathrm{T}}\right)^{-1}\kappa^{\mathrm{T}}}{W(t)}\left\{\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}}\left(\frac{1}{\gamma}\right)\left[1+G_1(v')-\mathrm{N}\left(-d_1(\underline{\zeta})\right)-Q\left(-d_1(\underline{\zeta}),v'\right)+\mathrm{N}\left(-d_1(\overline{\zeta})\right)+Q\left(-d_1(\overline{\zeta}),v'\right)\right]\right.$$

$$\left.-\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}\|\kappa\|\sqrt{T-t}}\left[-\phi\left(-d_1(\underline{\zeta})\right)\left(1+\varphi\left(\alpha(\underline{\zeta})\right)\right)+\phi\left(-d_1(\overline{\zeta})\right)\left(1+\varphi\left(\alpha(\overline{\zeta})\right)\right)\right]\right.$$

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$$+\frac{\underline{W}e^{-r(T-t)}}{\|\kappa\|\sqrt{T-t}}\left[-\phi\left(-d_2(\underline{\zeta})\right)\left(1+\varphi\left(\alpha(\underline{\zeta})\right)\right)+\phi\left(-d_2(\overline{\zeta})\right)\left(1+\varphi\left(\alpha(\overline{\zeta})\right)\right)\right]\Bigg\}$$

$$= \frac{\left(\sigma(t)^{\mathrm{T}}\right)^{-1}\kappa^{\mathrm{T}}}{W(t)}\Bigg\{\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{1/\gamma}}\left(\frac{1}{\gamma}\right)\left[1+G_1(v')-\mathrm{N}\left(-d_1(\underline{\zeta})\right)-Q\left(-d_1(\underline{\zeta}),v'\right)+\mathrm{N}\left(-d_1(\overline{\zeta})\right)+Q\left(-d_1(\overline{\zeta}),v'\right)\right]$$

$$-\frac{\underline{\underline{W}}e^{-r(T-t)}}{\|\kappa\|\sqrt{T-t}}\left[\phi\left(-d_2(\overline{\zeta})\right)\left(1+\varphi\left(\alpha(\overline{\zeta})\right)\right)\right]$$

$$+\frac{\underline{W}e^{-r(T-t)}}{\|\kappa\|\sqrt{T-t}}\left[\phi\left(-d_2(\overline{\zeta})\right)\left(1+\varphi\left(\alpha(\overline{\zeta})\right)\right)\right]\Bigg\}$$

$$= \frac{\left(\sigma(t)^{\mathrm{T}}\right)^{-1}\kappa^{\mathrm{T}}}{W(t)}\left(\frac{1}{\gamma}\right)\Bigg\{W(t)-\underline{W}e^{-r(T-t)}\left[\mathrm{N}\left(-d_2(\underline{\zeta})\right)+Q\left(-d_2(\underline{\zeta}),v\right)-\mathrm{N}\left(-d_2(\overline{\zeta})\right)-Q\left(-d_2(\overline{\zeta}),v\right)\right]$$

$$-\frac{(\underline{\underline{W}}-\underline{W})e^{-r(T-t)}}{\|\kappa\|\sqrt{T-t}}\left[\phi\left(-d_2(\overline{\zeta})\right)\left(1+\varphi\left(\alpha(\overline{\zeta})\right)\right)\right]\Bigg\}$$

where

$$\frac{e^{\Gamma(t)}}{\left(y\zeta(t)\right)^{\frac{1}{\gamma}}}e^{r(T-t)}\left[-\phi\left(-d_1(\underline{\zeta})\right)\left(1+\varphi\left(\alpha(\underline{\zeta})\right)\right)+\phi\left(-d_1(\overline{\zeta})\right)\left(1+\varphi\left(\alpha(\overline{\zeta})\right)\right)\right]$$

$$=\frac{e^{\frac{1}{\gamma}\left(r+\frac{\|\kappa\|^2}{2}\right)(T-t)+\left[\left(\frac{1}{\gamma}\right)^2-\frac{2}{\gamma}\right]\frac{\|\kappa\|^2}{2}(T-t)}}{\left(y\zeta(t)\right)^{\frac{1}{\gamma}}}\left[\frac{-1}{\sqrt{2\pi}}e^{-\frac{\left(d_2(\underline{\zeta})+\frac{1}{\gamma}v\right)^2}{2}}\left(1+\varphi\left(\alpha(\underline{\zeta})\right)\right)+\frac{1}{\sqrt{2\pi}}e^{-\frac{\left(d_2(\overline{\zeta})+\frac{1}{\gamma}v\right)^2}{2}}\left(1+\varphi\left(\alpha(\overline{\zeta})\right)\right)\right]$$

$$=\frac{e^{\frac{1}{\gamma}r(T-t)-\frac{1}{\gamma}\frac{\|\kappa\|^2}{2}(T-t)}}{\left(y\zeta(t)\right)^{\frac{1}{\gamma}}}\left[\frac{\frac{-1}{\sqrt{2\pi}}e^{-\frac{\left(d_2(\underline{\zeta})\right)^2}{2}}}{e^{\frac{1}{\gamma}vd_2(\underline{\zeta})}}\left(1+\varphi\left(\alpha(\underline{\zeta})\right)\right)+\frac{\frac{-1}{\sqrt{2\pi}}e^{-\frac{\left(d_2(\overline{\zeta})\right)^2}{2}}}{e^{\frac{1}{\gamma}vd_2(\overline{\zeta})}}\left(1+\varphi\left(\alpha(\overline{\zeta})\right)\right)\right]$$

$$=\frac{e^{\frac{1}{\gamma}\left(r-\frac{\|\kappa\|^2}{2}\right)(T-t)-\frac{1}{\gamma}vd_2(\underline{\zeta})}}{\left(y\zeta(t)\right)^{\frac{1}{\gamma}}}\left[-\phi\left(-d_2(\underline{\zeta})\right)\left(1+\varphi\left(\alpha(\underline{\zeta})\right)\right)\right]+\frac{e^{\frac{1}{\gamma}\left(r-\frac{\|\kappa\|^2}{2}\right)(T-t)-\frac{1}{\gamma}vd_2(\overline{\zeta})}}{\left(y\zeta(t)\right)^{\frac{1}{\gamma}}}\left[\phi\left(-d_2(\overline{\zeta})\right)\left(1+\varphi\left(\alpha(\overline{\zeta})\right)\right)\right]$$

$$\left(\because \frac{1}{\gamma}vd_2(\underline{\zeta})=\frac{1}{\gamma}\left[\ln\left(\frac{\underline{\zeta}}{\zeta(t)}\right)+\left(r-\frac{\|\kappa\|^2}{2}\right)(T-t)\right]\right)$$

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$$= \frac{-\left(\frac{\zeta(t)}{\underline{\zeta}}\right)^{\frac{1}{\gamma}}}{(y\zeta(t))^{\frac{1}{\gamma}}}\phi\left(-d_2(\underline{\zeta})\right)\left(1+\varphi\left(\alpha(\underline{\zeta})\right)\right)+\frac{\left(\frac{\zeta(t)}{\overline{\zeta}}\right)^{\frac{1}{\gamma}}}{(y\zeta(t))^{\frac{1}{\gamma}}}\phi\left(-d_2(\overline{\zeta})\right)\left(1+\varphi\left(\alpha(\overline{\zeta})\right)\right)$$

$$= -(y\underline{\zeta})^{\frac{-1}{\gamma}}\phi\left(-d_2(\underline{\zeta})\right)\left(1+\varphi\left(\alpha(\underline{\zeta})\right)\right)+(y\overline{\zeta})^{\frac{-1}{\gamma}}\phi\left(-d_2(\overline{\zeta})\right)\left(1+\varphi\left(\alpha(\overline{\zeta})\right)\right)$$

$$= -\underline{W}\phi\left(-d_2(\underline{\zeta})\right)\left(1+\varphi\left(\alpha(\underline{\zeta})\right)\right)+\underline{\underline{W}}\phi\left(-d_2(\overline{\zeta})\right)\left(1+\varphi\left(\alpha(\overline{\zeta})\right)\right)$$

$\because \theta^{VaR}(t) = q^{VaR}(t)\theta^{B}(t)$ and $\theta^{B}(t) = \frac{1}{\gamma}(\sigma(t)^{\mathrm{T}})^{-1}\kappa$

$$\therefore q^{VaR}(t) = 1 - \frac{\underline{\underline{W}}e^{-r(T-r)}\left\{\left[\mathrm{N}\left(-d_2(\underline{\zeta})\right)-\mathrm{N}\left(-d_2(\overline{\zeta})\right)\right]+\left[Q\left(-d_2(\underline{\zeta}),v\right)-Q\left(-d_2(\overline{\zeta}),v\right)\right]\right\}}{W(t)}$$

$$+\frac{\gamma e^{-r(T-t)}\phi\left(d_2(\overline{\zeta})\right)(\underline{W}-\underline{\underline{W}})\left[1+\varphi\left(\alpha(\overline{\zeta})\right)\right]}{W(t)\|\kappa\|\sqrt{T-t}}$$

where $\varphi(\alpha(x)) = \beta\left[\alpha^3(x)-3\alpha(x)\right]+\delta\left[\alpha^4(x)-6\alpha^2(x)+3\right]$

$$\alpha(x) = \frac{\ln(x)-m}{v} = d_1(x)+v' = d_2(x)+v$$

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